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Statistical analysis of the Hertzian fracture of pyrex glass using the Weibull distribution function

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Hertzian fracture tests were carried out on specimens of ground-and-polished Pyrex glass using polished Pyrex glass balls of 6 and 8 mm diameter. The results were analysed according to the theory of flaw statistics originally proposed by Weibull. The Weibull parameters m and σ_0 were found to be independent of ball size; σ_u however decreased with increase in ball size. The parameters σ_u, σ_0 and m obtained from the Hertzian tests differed from those obtained from a four-point bend test. The predicted mean fracture stress and the mean fracture location for Hertzian fracture using the derived Weibull parameters agreed reasonably well with the experimental values.

1. Introduction

In a previous paper [1] on the Hertzian fracture of Pyrex glass it was found that the critical load to cause Hertzian fracture exhibited appreciable scatter. The scatter was found to increase with indenter size and decrease upon abrading the as-received (ground-and-polished) glass specimen surface. Also, other surface treatments like annealing or etching in HF caused an increase in the scatter. The present paper examines the statistical nature of the scatter in terms of the Weibull flaw distribution function [2] often employed to describe the fracture strength of brittle solids.

It is generally recognized that the fracture of a brittle material is largely determined by the number and severity of flaws on its surface. This results in considerable scatter in the fracture stress, which depends on the flaw distribution. Weibull [2] first analysed the statistical nature of the fracture behaviour of brittle materials using a distribution function to characterize the flaws on the surface. A number of investigators [3–6] subsequently modified Weibull's approach for determining the flaw distribution function from strength measurements. However, there still exists a lack of agreement as to the validity of these analyses when applied to the Hertzian fracture of glass [6–13]. The more recent work of Oh and Finnie [12] and Hamilton and Rawson [13] on Hertzian fracture of glass using the Weibull distribution function explains some of the experimental observations. However, some of the assumptions in their analyses may be questioned. It is therefore the purpose of this paper to re-examine the application of the Weibull theory to the Hertzian fracture of Pyrex glass.

2. Theoretical considerations

Many investigators have used the original flaw distribution function proposed by Weibull [2]

$$n(\sigma) = \int_0^{\sigma} \phi(s)\, \mathrm{d}s \qquad (1)$$

where $n(\sigma)$ is the total number of flaws per unit area of surface which can act as fracture origins when the applied stress is σ and $\phi(s)$ is the distribution function of crack sizes. Now suppose that A is the surface area of a specimen having a surface characterized by a flaw distribution $n(\sigma)$ and that the specimen is loaded in such a way that the stress is uniform over the whole surface of the specimen. It can then be shown that the fraction of specimens fracturing at a stress in the range 0 to σ is given by

$$F(\sigma) = 1 - \exp\,[-n(\sigma)A] \qquad (2)$$

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where F is the cumulative probability of fracture for this range of stress. However, when a spherical indenter is pressed onto the surface of a specimen, as in a Hertzian fracture test, the stress distribution on the surface is non-uniform. The stress varies with distance according to

$$\sigma_{rr} = \sigma_a(a/r)^2 \quad (3)$$

where σ_a and σ_{rr} are the stress values at the contact radius a and at the radial distance r, respectively. In this case, Oh and Finnie [12] have shown that the cumulative probability of fracture $F(\sigma)$ is given by

$$F(\sigma) = 1 - \exp\left[-\int_a^\infty n(\sigma) 2\pi r \, dr\right] \quad (4)$$

Weibull [2] and other investigators [12, 13] assume that $n(\sigma)$ is of the form

$$n(\sigma) = \left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m, \quad \sigma > \sigma_u \quad (5)$$

$$= 0, \qquad \sigma < \sigma_u \quad (5a)$$

The parameters σ_u, σ_0 and m are constants for a particular surface and it is desired to find best values which fit the experimental results. Hamilton and Rawson [13] assume that σ_u, σ_0 and m are independent of ball size in Hertzian fracture tests. However, the results of Conrad *et al.* [1] suggest that the zero probability stress σ_u is dependent on ball size. Further, Oh and Finnie [12] use the parameter obtained from four-point bend tests to analyse the Hertzian fracture behaviour of glass. However, it will be shown below that the parameters σ_u, σ_0 and m determined from bend tests differ from those of Hertzian fracture tests. A similar conclusion was reached earlier by Lewis and Rawson [14]. Hence, one cannot use parameters determined from bend tests to analyse Hertzian fracture test results.

Finally, in the earlier paper [1] it was shown that the stress distribution on the surface of Hertzian fracture test specimens is altered by friction between the indenter and the specimen and by surface roughness. The stress variation due to these factors was calculated by Johnson *et al.* [15] and shown to be complex. Hence, to minimize roughness and friction effects in the present study, polished Pyrex glass spheres were employed as indenters, which were loaded onto the ground-and-polished Pyrex glass specimens.

3. Materials and test procedure

3.1. Hertzian fracture tests

The specimen and indenter materials used in the Hertzian fracture tests and the test procedure are described in detail in [1]. Briefly, the specimen material was 10 mm thick Corning Pyrex 7740 glass plate (ground-and-polished). The present Hertzian fracture tests consisted of pressing 6 and 8 mm diameter Pyrex glass balls onto the specimen surface using an Instron tensile testing machine at a cross-head velocity of 8.5×10^{-6} m sec^{-1}.

3.2. Bend tests

The standard ASTM four-point bend test procedure [16] was followed. The specimens (10 cm × 1.25 cm × 1 cm thick) were cut from ground-and-polished Corning Pyrex 7740 glass plates. The bend tests were conducted using an Instron tensile testing machine at the same cross-head velocity as the Hertzian fracture tests. The fracture samples were examined visually and the fracture stress was taken from those samples which failed due to the surface flaws and not edge flaws.

4. Determination of the Weibull parameters from the experimental results

4.1. Bend tests

Weil and Daniel [17] have shown for a material which fails due to a distribution of flaws on its surface that the exponential in Equation 4 may be integrated (using $n(\sigma)$ defined by Equation 5) for pure bending of a specimen with rectangular cross-section to give

$$F(\sigma) = 1 - \exp\left[bl\left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m - \frac{h}{(m+1)} \times \left(1 - \frac{\sigma_u}{\sigma_0}\right)\left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m\right] \quad (6)$$

where b, h and l are width, thickness and length of the specimen, respectively. Taking $F(\sigma) = n/(N+1)$, where n is the number assigned to a specimen when the N specimens are ranked from 1 to N in order of increasing fracture strength, and substituting the value of $F(\sigma)$ into Equation 6, one obtains upon taking the logarithm twice

$$\log\log\left(\frac{N+1}{N+1-n}\right) - \log\left[1 + \frac{h}{b(m+1)}(1 - \sigma_u/\sigma_0)\right] = m\log(\sigma - \sigma_u) + \log\frac{bl}{\sigma_0 m} + \log\log(e). \quad (6a)$$

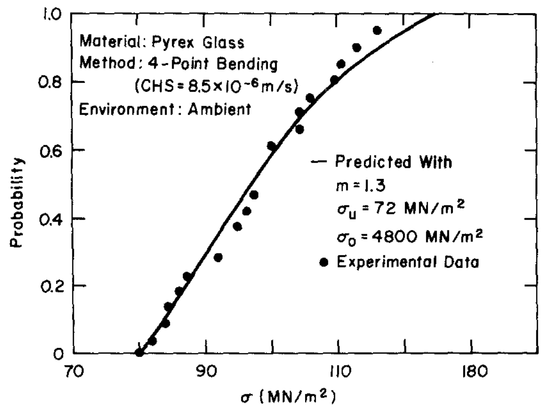

Figure 1 Comparison of the experimental values of probability versus fracture stress from bend tests with those calculated using the Weibull distribution function.

The parameters σ_u, σ_0 and m were determined from the bend tests using Equation 6 in the following manner. Assuming that the contribution due to the second term on the left-hand side of Equation 6a to be small, plots of $\log\log(N+1)/(N+1-n)$ as a function of $\log(\sigma - \sigma_u)$ for a selected series of σ_u were drawn. The slope of the straight line passed through each set of data yields to a first approximation the value of m for each value of σ_u. The best value of m was selected, and again using Equation 6a the left-hand side was plotted versus the right-hand side for various σ_u values until a good correlation was obtained. Fig. 1 shows the actual experimental data points and the predicted curve based on the procedure just described. Table I lists the best-fit values of σ_u, σ_0 and m for the bend tests on the present material. Also included are the values of σ_u, σ_0 and m obtained by Oh and Finnie [12] for Pyrex glass.

4.2. Hertzian fracture tests

Oh and Finnie [12] showed that the probability of fracture in a Hertzian fracture test is given by

$$F(\sigma) = 1 - \exp - \int_0^{r_u} \left(\frac{\sigma_{rr} - \sigma_u}{\sigma_0}\right)^m 2\pi r \, dr \tag{7}$$

r_u being the value of the radial distance r at which $\sigma_{rr} = \sigma_u$. Hamilton and Rawson [13] simplified Equation 7 using the Hertzian equation for contact stresses, namely

$$\sigma_{rr} \begin{cases} < 0 & 0 \leqslant r < a \\ = \sigma_a (a/r)^2 & r \geqslant a \end{cases} \tag{8}$$

where $a^3 = K_1 PR$ and $a = K\sigma_a$. P is the load and R is the ball radius. Constants K_1 and K are given by

$$K_1 = \frac{3}{4}\left[\frac{1-\nu'^2}{E'} + \frac{1-\nu^2}{E}\right] \tag{9}$$

$$K = \frac{2\pi K_1 R}{1 - 2\nu} \tag{10}$$

Here ν', E' and ν, E are Poisson's ratio and Young's modulus of the ball and specimen respectively. Equation 7 can then be written as

$$F(\sigma) = 1 - \exp\left[-I(\sigma_a)\right] \tag{7a}$$

where $I(\sigma_a)$ is given by

$$I(\sigma_a) = \pi a^2 \left(\frac{\sigma_a - \sigma_u}{\sigma_0}\right)^m \times \left[\frac{(1-x)}{1+m} + \frac{2!\,(1-x)^2}{(1+m)(3+m)} + \ldots\right] \tag{11}$$

and

$$x = (\sigma_u/\sigma_a).$$

TABLE I Weibull parameters σ_u, σ_0 and m for the fracture of Corning 7740 Pyrex glass in bend tests

Surface condition	Present work as-received (ground-and-polished)	Oh and Finnie [12]* (surface not specified)
$\sigma_u (\mathrm{N\,m^{-2}})$	7.18×10^7	2.8×10^7 3.5×10^7
$\sigma_0 (\mathrm{N\,m^{-2}})$	4.80×10^9	3.1×10^7 2.4×10^7
m	1.3	4.2 3.2

* Two values of each of the parameters σ_u, σ_0 and m are given here for the results by Oh and Finnie [12], because it was difficult to select unique values from their graphs.

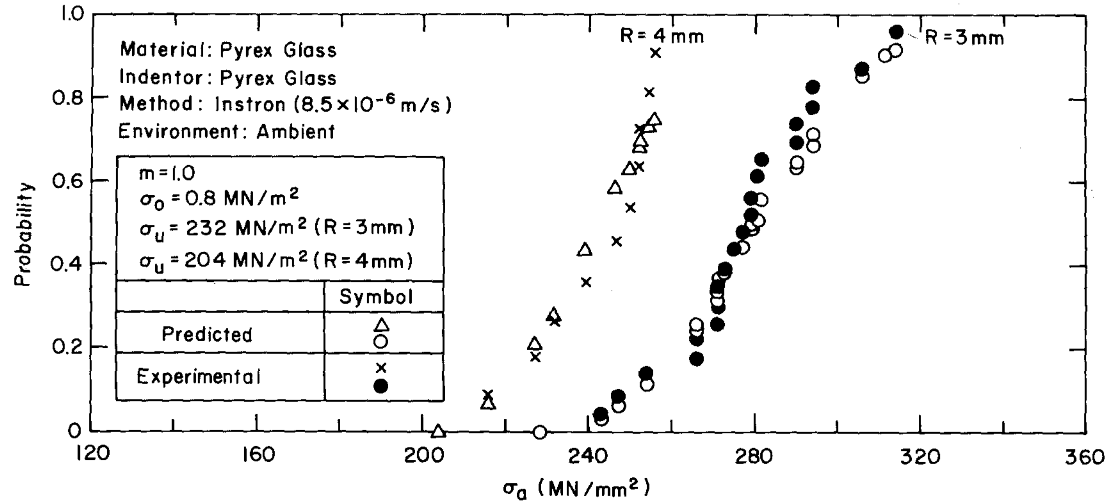

Figure 2 Comparison of the experimental values of probability versus fracture stress from Hertzian fracture tests with those calculated using the Weibull distribution function.

Taking logarithms twice, Equation 11 becomes

$$\log\log\left(\frac{1}{1-F}\right) = \log I(\sigma_a) \quad (11a)$$

$$\log\log\left(\frac{1}{1-F}\right) = \log\left\{(\sigma_a-\sigma_u)^m a^2\left[\frac{(1-x)}{(1-m)} + \frac{2!\,(1-x)^2}{(1+m)(3+m)} + \ldots\right]\right\} + \log(\pi/\sigma_0^m) \quad (11b)$$

Equation 11b is of the form

$$Y = MX + \text{constant}. \quad (12)$$

The Weibull parameters, σ_u, σ_0 and m for the present Hertzian tests were determined using Equation 11b as follows. An approximate value of σ_u is selected first from the experimental data by extrapolation to zero probability of fracture. Using the selected experimental values of σ_a, F and σ_u, the values of the left-hand side of Equation 11b are plotted against the right-hand side (neglecting the second term) for the various values of m. The best value of m corresponds to that when the slope of the plot is equal to 1.0. The intercept gives the value of $\log(\pi/\sigma_0^m)$. The m value so obtained was 1.0 and σ_0 was 0.80 MN m^{-2}. Taking these values of σ_0 and m and the experimental values of σ_a and F, the best value of σ_u is found.

The values of σ_u, σ_0 and m so determined are listed in Table II. Fig. 2 compares the experimental values of σ_a with the calculated values using the derived Weibull parameters for the two ball sizes. The agreement between the two is good.

5. Discussion

The parameters σ_u, σ_0 and m obtained by the Hertzian fracture tests differ from those by the four-point bend tests. Hence, one cannot use the parameters determined from the bend tests in Hertzian fracture analysis. Further, the parameter σ_u was found to be ball size dependent. It was found in the earlier work [1] that the stress at 50% probability varies as some power of the ball size. Since σ_u is here defined as the zero probability

TABLE II Comparison of Weibull parameters obtained from the four-point bend tests and the Hertzian fracture tests for ground-and-polished Pyrex glass

Weibull parameters	Four-point bend tests	Hertzian tests	
		3 mm*	4 mm*
σ_u(N m^{-2})	7.18 × 10^7	2.32 × 10^8	2.04 × 10^8
σ_0(N m^{-2})	4.80 × 10^9	8.00 × 10^5	8.00 × 10^5
m	1.3	1.0	1.0

* Pyrex glass ball radius.

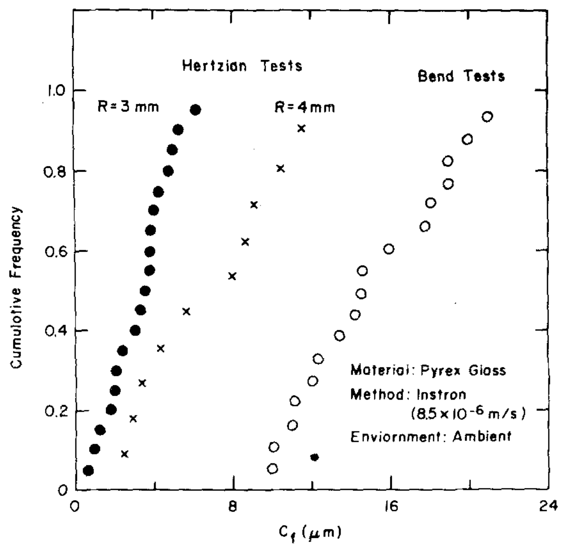

Figure 3 Cumulative frequency (probability) versus the flaw size c_f calculated from bend test and Hertzian fracture test results for Pyrex glass.

stress, it is not surprising that it is also dependent on the ball size.

It was proposed in the earlier paper [1] that the flaw size distribution causing the brittle fracture in both bend tests and Hertzian fracture tests can be determined using

$$c_f = \left\{ \frac{2E\gamma_s}{(1-\nu^2)\pi} \right\} \frac{1}{(1.12\sigma_f)^2} \qquad (13)$$

Where c_f is the flaw size, γ_s, ν and E are respectively surface energy, Poisson's ratio and Young's modulus of the specimen and σ_f is the fracture stress. Fig. 3 shows the flaw size distribution calculated from Hertzian fracture and the bend test results using Equation 13. It is clear that the flaw sizes involved in the fracture are different for the two test methods. This provides further evidence that one cannot use the Weibull parameters obtained from bend tests to analyse Hertzian fracture. Also, as mentioned above, one cannot assume σ_u to be ball size independent.

Table III gives the mean fracture stress and the mean fracture location determined using the Weibull parameters from Hertzian fracture tests. The mathematical analysis is similar to that by Oh and Finnie [12]. Reasonably good agreement exists between the experimental and predicted values. Also, it is clear that the flaw distribution can account for the ratio of the size of the crack radius r^* to the contact radius a (i.e. r^*/a) to have a value up to 1.1. However, for a rough surface and for indenters with appreciably different elastic constants than the specimen friction comes into play due to elastic mismatch and roughness, which alter the stress distribution to give r^*/a ratios up to 1.5 [1]. Thus, one cannot account for these larger r^*/a ratios from flaw distribution considerations alone. A similar conclusion was reached by Johnson *et al.* [15].

6. Conclusions

(1) The Weibull distribution function satisfactorily characterizes the fracture strength of Pyrex glass in four-point bend and Hertzian fracture tests. The Weibull parameters σ_u, σ_0 and m determined for a given surface condition from a four-point bend test and a Hertzian fracture test are different. Similar conclusions were reached by Lewis and Rawson [14].

(2) The Weibull parameters σ_0 and m determined from a Hertzian fracture test are independent of ball size. However the zero probability stress σ_u is ball size dependent, similar to the 50% probability or mean fracture stress.

(3) The mean fracture stress and mean fracture location predicted using the Weibull distribution function are in reasonable agreement with the experimental values when the elastic mismatch between the indenter and the specimen surface are the same and the surfaces are relatively smooth. Also, the existing flaw distribution can account for the ratio of ring crack radius to contact radius $r^*/a < 1.1$. However, friction effects due to surface roughness and elastic mismatch can displace the crack location to considerably larger r^*/a values.

TABLE III Comparison of the mean fracture stress (σ_f), mean fracture location (r^*) and contact radius (a) predicted using Weibull parameters with the experimental values obtained from the Hertzian fracture of Pyrex glass

Pyrex glass ball radius (mm)	Experimental				Predicted			
	a (mm)	r^* (mm)	σ_f (MN m^{-2})	(r^*/a)	a (mm)	r^* (mm)	σ_f (MN m^{-2})	(r^*/a)
3	0.19	0.20	279	1.05	0.20	0.21	275	1.10
4	0.23	0.27	243	1.12	0.23	0.24	239	1.09

Hence, in order to analyse statistically the results from the Hertzian test using the Hertzian stress distribution and the Weibull distribution function, it is important that the indenter and specimen have the same elastic constants and also that the surface of both are very smooth.

(4) The flaw size distribution causing Hertzian fracture varies with the ball size. Also, the flaws causing fracture in four-point bending are different (larger flaws) than those in Hertzian fracture. Hence, to characterize fully the flaw size distribution of a surface by Hertzian fracture tests one needs to use a wide range of ball sizes.

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